Mathematical Physics

Divergence of a vector :-

del dot $\vec{V} = \nabla . \vec{V}$

$$\vec{\nabla}.\vec{V} = \left(\hat{i}\, \partial/\partial x + \hat{j}\, \partial/\partial y + \hat{k}\, \partial/\partial z\right) . \left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k}\right)$$

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

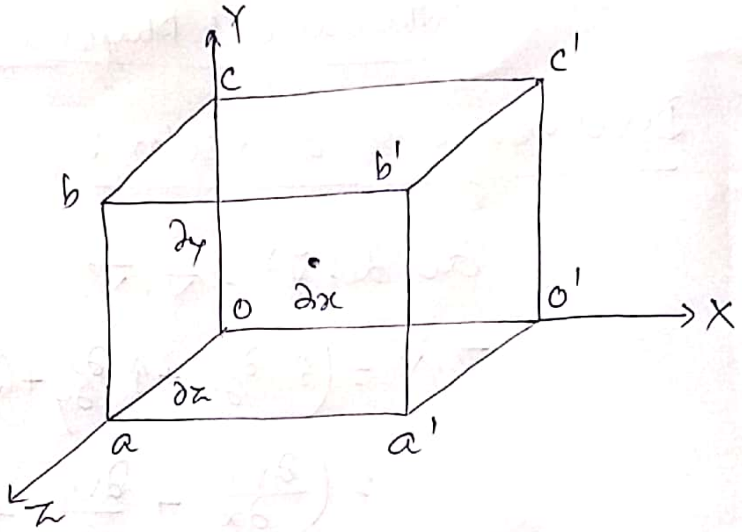
Here, $\vec{\nabla}$ is a vector differential operator But, $\nabla.\vec{V}$ is a scalar function. An alternative difinition of defenition of a divergence of a vector. Suppose that there is a vector field $\vec{V}$ and there is a cuboid in this vector field, then x- component of vector at P point is V_x.

So, the value of x – component of $\vec{V}$ on a'o'c'b' surface will be $V_x + \partial V_x / \partial x \, \partial x/2$ On oabc surface, the x- component of vector will be $(V_x - \partial V_x / \partial x \, \partial x/2)$.

Hence, the flux entering the oabc surface $= (V_x - \partial V_x/\partial x \, \partial x/2) \partial y \, \partial z$

and the flux emitting from o'a'b'c' surface $= (V_x + \partial V_x/\partial x + \partial x/2) dy\, dz$

the net flux emitting from surface $\perp r$. to x-axis $= \partial V_x/\partial x \cdot \partial x \partial y \partial z$ —— (1)



Similarly, flux emitted from the surface perpendicular to Y-axis

$$= \partial V_y/\partial y \, \partial x \partial y \partial z$$

and the flux emitting from the surface perpendicular to z-axis

$$= \partial V_z/\partial z \, \partial x \partial y \partial z.$$

Therefore, total flux emitted from the cuboid

$$= \left(\partial V_x/\partial x + \partial V_y/\partial y + \partial V_z/\partial z\right) \partial x \partial y \partial z$$

Hence, flux emitted from unit volume

$$= \left(\partial V_x/\partial x + \partial V_y/\partial y + \partial V_z/\partial z\right) = \nabla \cdot \vec{V}$$

Hence, divergence of a vector is equal to the total flux emitted from unit volume of a cuboid.

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Curl of a vector $\nabla \times \vec{V}$ = curl $\vec{V}$:-

$$\vec{\nabla} \times \vec{V} = (\hat{i}\, \partial/\partial x + \hat{j}\, \partial/\partial y + \hat{k}\, \partial/\partial z) \times (V_x \hat{i} + V_y \hat{j} + V_z \hat{k})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & \partial z \end{vmatrix}$$

Let us consider a rectangular surface abcd which is perpendicular to z-axis the vector function at origine is $\vec{V}$ its x-component is V_x and y-component is V_y at one origine, then x-component of vector on ab line will be

$$\left(V_x - \partial V_y/\partial y \; \partial y/2\right)$$

So,

$$\int_a^b \vec{V} \cdot d\vec{l} = \left(V_x - \partial V_x/\partial y \; \partial y/2\right) \partial x \quad -(1)$$

where, ab = ∂x

the y-component of vector field at b

$$= V_y + \partial V_y/\partial x \cdot \partial x/2$$

So,

$$\int_b^c \vec{V} \cdot d\vec{l} = \left(V_y + \partial V_y/\partial x \cdot \partial x/2\right) \partial y$$

$$\int_c^d \vec{V}\cdot\vec{dl} = -(V_x + \partial V_x/\partial y \;\partial y/2)\,\partial x$$

$$\int_d^a \vec{V}\cdot\vec{dl} = -(V_y - \partial V_y/\partial x \;\partial x/2)\cdot\partial y$$

$$\therefore \int_{abcda}^{a} \vec{V}\cdot\vec{dl} = V_x\partial x - \partial V_x/\partial y \;\partial x\partial y/2 + V_y\partial y + \partial V_y/\partial x \cdot \partial x\partial y/2$$

$$- \cancel{V_x\partial x} - \partial V_x/\partial y \cdot \partial x\,\partial y/2 - V_y\partial y + \partial V_y \cdot \partial x\partial y$$

$$= \partial V_y/\partial x \cdot \partial x\partial y - \partial V_x/\partial y \cdot \partial x\partial y$$

$$\text{curl}\,\vec{V} = 1/\partial x\partial y \int_{abcda} \vec{V}\cdot\vec{dl}$$

$$= \left(\partial V_y/\partial x - \partial V_x/\partial y\right)\hat{k}$$

Similarly, x and y component of curl $\vec{V}$ can be calculated.